the stock should not be purchased at the close on Day 50. The smoothed version of $c[n]$ can be recomputed as new values for $c[n]$ become available. A buy signal would be generated when the slope of $\hat{c}[n]$ becomes positive. A sell signal would be generated if the slope of $\hat{c}[n]$ has been positive and then becomes negative. This approach to trading does not always work well, since as noted previously, the slope of $\hat{c}[n]$ may not be an accurate indication of the trend of $c[n]$ at $n = N - 1$. The interested reader should practice this strategy extensively, using past data before attempting to use it for actual trading.

4.6 CHAPTER SUMMARY

The discrete-time Fourier transform (DTFT) is used to determine the frequency content of discrete-time signals. The DTFT of a signal $x[n]$ is a function $X(\Omega)$ of the frequency variable Ω defined as

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

The resulting DTFT is periodic with period 2π. For a real-valued signal $x[n]$, the magnitude function $|X(\Omega)|$ is an even function of Ω, and the angle function $\angle X(\Omega)$ is an odd function of Ω. Thus, the frequency spectrum can be plotted only over the interval $0 \le \Omega \le \pi$. The frequency shown in some of the plots is normalized by π, resulting in a frequency range of 0 to 1. Useful properties of the DTFT include linearity, left and right shifts in time, multiplication by a sinusoid, and convolution in the time domain.

When the discrete-time signal $x[n]$ is time-limited, that is, $x[n] = 0$ for $n < 0$ and $n \ge N$ for some positive integer N, the discrete Fourier transform (DFT) is equal to the DTFT at the discrete frequencies $\Omega = 2\pi k/N$, with the values of the transform given by

$$X_k = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots, N-1$$

The DFT is often used in place of the DTFT when the frequency content of a signal $x[n]$ needs to be computed numerically. This computation is actually an approximation of the DTFT when the signal is not time-limited and needs to be truncated in order to perform the DFT. The approximation errors can be made small if the values of the truncated part of the signal are negligible, or if the signal is periodic and the length of the truncated signal is an integer multiple of the period.

The DFT is very useful for numerical analysis of continuous-time data that has been recorded digitally by signal sampling. The FFT, a fast algorithm for computing the DFT, is used commonly in engineering applications where signal processing of measured data is performed. In these cases, the engineer or scientist is actually interested in the frequency content of a continuous-time signal. The FFT of the sampled signal can be used to approximate the continuous-time Fourier transform of the signal. The approximation improves if the sampling period T is decreased, and the resolution in the plot improves if the product NT is increased where N corresponds to the number of points in the data record. If the signal is periodic, then leakage in the spike about the fundamental frequency can be lessened by making N equal to an integer multiple of the fundamental period of the signal. This is illustrated in the example involving sunspot data in Section 4.5. In this section, it is also shown that the DFT (or FFT) can be used to extract the dominant cyclic components of a signal for the purpose of carrying out signal analysis.

PROBLEMS

4.1. Compute the DTFTs of the discrete-time signals shown in Figure P4.1. Express the DTFTs in the simplest possible form. Plot the amplitude and phase spectrum for each signal.

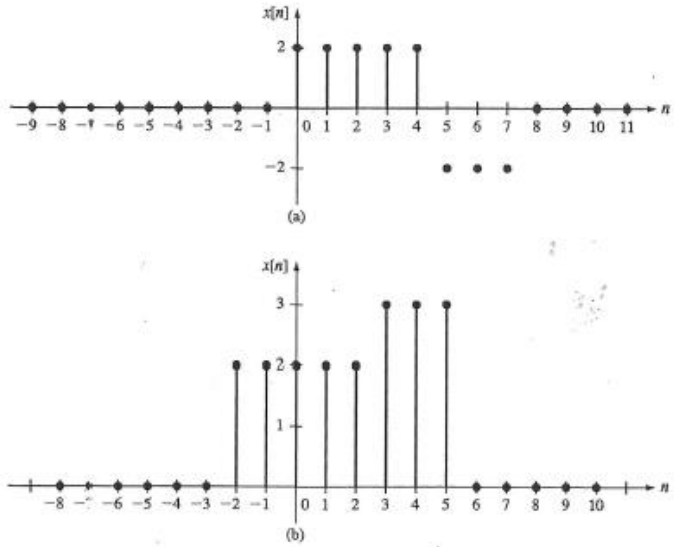


FIGURE P4.1

4.2. Prove the following relationship:

$$\sum_{n=q_1}^{q_2} r^n = \frac{r^{q_1} - r^{q_2+1}}{1 - r}$$

Hint: Multiply both sides of the equation by $(1 - r)$.

4.3. By breaking the DTFT $X(\Omega)$ into three summations, (from $n = -\infty$ to $n = -1$; $n = 0$; and $n = 1$ to $n = \infty$) and applying Euler's formula, prove the following:

(a) If $x[n]$ is an even function of n, then $X(\Omega) = x[0] + \sum_{n=1}^{\infty} 2x[n] \cos \Omega n$.

(b) If $x[n]$ is an odd function of n, then $X(\Omega) = x[0] - \sum_{n=1}^{\infty} j2x[n] \sin \Omega n$.

4.4. Compute the DTFT of the following discrete-time signals; plot the amplitude and the phase spectrum for each signal:

(a) $x[n] = (0.8)^n u[n]$

(b) $x[n] = (0.5)^n \cos 4n\, u[n]$

(c) $x[n] = n(0.5)^n u[n]$

(d) $x[n] = n(0.5)^n \cos 4n\, u[n]$

(e) $x[n] = 5(0.8)^n \cos 2n\, u[n]$

(f) $x[n] = (0.5)^{|n|}, -\infty < n < \infty$

(g) $x[n] = (0.5)^{|n|} \cos 4n, -\infty < n < \infty$

4.5. A discrete-time signal $x[n]$ has DTFT

$$X(\Omega) = \frac{1}{e^{j\Omega} + b}$$

where b is an arbitrary constant. Determine the DTFT $V(\Omega)$ of the following:

(a) $v[n] = x[n - 5]$

(b) $v[n] = x[-n]$

(c) $v[n] = nx[n]$

(d) $v[n] = x[n] - x[n - 1]$

(e) $v[n] = x[n] * x[n]$

(f) $v[n] = x[n] \cos 3n$

(g) $v[n] = x^2[n]$

(h) $v[n] = x[n]e^{j2n}$

4.6. Use (4.28) or the properties of the DTFT to compute the inverse DTFT of the frequency functions $X(\Omega)$ shown in Figure P4.6.

4.7. Determine the inverse DTFT of the following frequency functions:

(a) $X(\Omega) = \sin \Omega$

(b) $X(\Omega) = \cos \Omega$

(c) $X(\Omega) = \cos^2 \Omega$

(d) $X(\Omega) = \sin \Omega \cos \Omega$

4.8. The autocorrelation of a discrete-time signal $x[n]$ is defined by

$$R_x[n] = \sum_{i=-\infty}^{\infty} x[i]x[n + i]$$

Let $P_x(\Omega)$ denote the DTFT of $R_x[n]$.

(a) Derive an expression for $P_x(\Omega)$ in terms of the DTFT $X(\Omega)$ of $x[n]$.

(b) Derive an expression for $R_x[-n]$ in terms of $R_x[n]$.

(c) Express $P_x(0)$ in terms of $x[n]$.

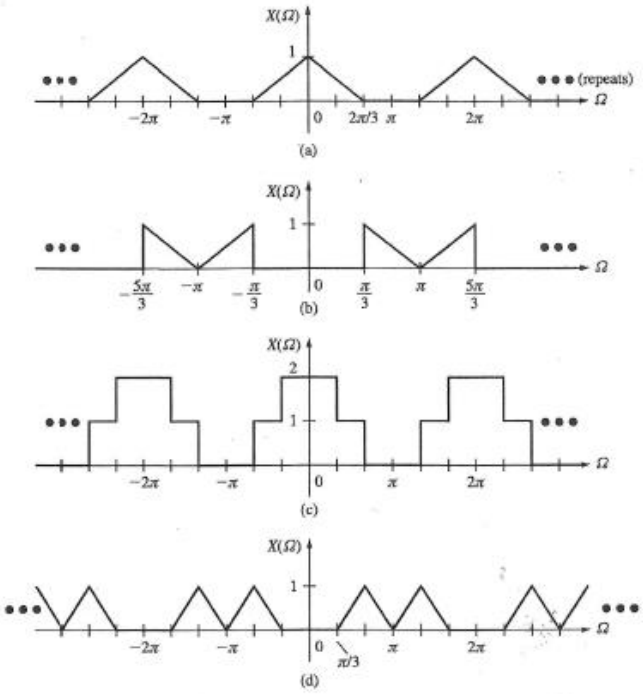


FIGURE I4.6

4.9. Compute the rectangular form of the four-point DFT of the following signals, all of which are zero for $n < 0$ and $n \geq 4$:

(a) $x[0] = 1, x[1] = 0, x[2] = 1, x[3] = 0$

(b) $x[0] = 1, x[1] = 0, x[2] = -1, x[3] = 0$

(c) $x[0] = 1, x[1] = 1, x[2] = -1, x[3] = -1$

(d) $x[0] = -1, x[1] = 1, x[2] = 1, x[3] = 1$

(e) $x[0] = -1, x[1] = 0, x[2] = 1, x[3] = 2$

(f) $x[0] = 1, x[1] = -1, x[2] = 1, x[3] = -1$

(g) Compute the DFT for each of the foregoing signals using the MATLAB M-file `dft`. Compare these results with the results obtained analytically in parts (a) to (f).

4.10. Verify the relationship given in Equation (4.42).

4.11. Derive the sinusoidal form (4.45) in the case when N is even.

4.12. Compute the sinusoidal form of the signals in Problem 4.9.

4.13. Using the MATLAB M-file `dft`, compute the 32-point DFT of the signals (a)–(f). Express your answer by plotting the amplitude $|X_k|$ and phase $\angle X_k$ of the DFTs.

(a) $x[n] = 1, 0 \le n \le 10, x[n] = 0$ for all other n

(b) $x[n] = 1, 0 \le n \le 10, x[n] = -1, 11 \le n \le 20, x[n] = 0$ for all other n

(c) $x[n] = n, 0 \le n \le 20, x[n] = 0$ for all other n

(d) $x[n] = n, 0 \le n \le 10, x[n] = 20 - n, 11 \le n \le 20, x[n] = 0$ for all other n

(e) $x[n] = \cos(10\pi n/11), 0 \le n \le 10, x[n] = 0$ for all other n

(f) $x[n] = \cos(9\pi n/11), 0 \le n \le 10, x[n] = 0$ for all other n

4.14. Using the MATLAB M-file `dft`, compute the magnitude of the 32-point DFT X_k of the following signals:

(a) $$x[n] = \begin{cases} 1, & n = 0 \\ \dfrac{1}{n}, & n = 1, 2, 3, \ldots, 31, \\ 0, & n = 32, 33, \ldots \end{cases}$$

(b) $$x[n] = \begin{cases} 1, & n = 0 \\ \dfrac{1}{n^2}, & n = 1, 2, 3, \ldots, 31 \\ 0, & n = 32, 33, \ldots \end{cases}$$

(c) $$x[n] = \begin{cases} 1, & n = 0 \\ \dfrac{1}{n!}, & n = 1, 2, 3, \ldots, 31 \\ 0, & n = 32, 33, \ldots \end{cases}$$

(d) Compare the results obtained for parts (a) to (c). Explain the differences in the results.

4.15. Consider the discrete-time signal

$$x[n] = \begin{cases} r[n] - 0.5, & n = 0, 1, 2, \ldots, 31 = N - 1 \\ 0, & \text{all other } n \end{cases}$$

where r is a sequence of random numbers uniformly distributed between 0 and 1. This sequence can be generated by the MATLAB command `rand(N,1)`. The signal $x[n]$ can be interpreted as random noise. Using the `dft` M-file, compute the magnitude of the 32-point DFT of $x[n]$. What frequencies would you expect to see in the amplitude spectrum of $x[n]$? Explain.

4.16. Use the MATLAB M-file `dft` with $N = 10$ to approximate the DTFT of the signal plotted in Figure P4.1a. Plot the amplitude and phase spectrum for X_k versus $\Omega = 2\pi k/N$. Compare this result to the DTFT obtained in Problem 4.1 over the frequency range $\Omega = 0$ to $\Omega = 2\pi$. Repeat for $N = 20$.

4.17. Repeat Problem 4.16 for the signal plotted in Figure P4.1b.

4.18. To determine the effect of truncation in computing the approximation of a DTFT by a DFT, consider the signal defined by $x[n] = n(0.5)^n u[n]$.

(a) Determine the minimum value of N so that the signal has magnitude $|x[n]| \le 20\%$ of its maximum value for all $n \ge N$.

(b) Use MATLAB to compute the 50-point DFT of the truncated signal $\tilde{x}[n]$ defined by

$$\tilde{x}[n] = \begin{cases} x[n], & 0 \le n \le N - 1 \\ 0, & \text{all other } n \end{cases}$$

where N was determined in part (a). Plot the amplitude and phase spectrum of X_k versus $\Omega = 2\pi k/50$.

(c) Compare the result obtained in part (b) with the DTFT of $x[n]$ found in Problem 4.4(c).

(d) Repeat parts (a) to (c) for the value of N such that the signal has magnitude $|x[n]| \le 5\%$ of its maximum value for all $n \ge N$.

4.19. This problem explores the use of the FFT in approximating the Fourier Transform of continuous-time signals.

(a) Compute the Fourier transform of $x(t) = 4e^{-4t}u(t)$.

(b) Create a sampled version of the signal $x(t)$ in MATLAB for the cases (i)–(iv), where T is the sampling time and N is the total number of points. Use the M-file `contfft.m` from the textbook to compute the approximation to $X(\omega)$. Plot $|X(\omega)|$ versus ω for the exact Fourier transform obtained in part (a) and for the approximated Fourier transform obtained from `contfft`, both on the same graph. (Use the range $0 \le \omega \le 50$ rad/sec.)

(i) $T = 0.5, N = 10$

(ii) $T = 0.1, N = 50$

(iii) $T = 0.05, N = 100$

(iv) $T = 0.05, N = 400$

(c) Identify the trends in accuracy and resolution in the plots, as T is decreased and as NT is increased.

4.20. Let $spd[n]$ denote the monthly sunspot data for the 33-year period from January 1875 through December 1907. Note that there are a total of $(12)(33) = 396$ data points.

(a) Download the data from the Web, using the procedure described in Section 4.5, and generate a MATLAB plot of $spd[n]$ for the 33-year period.

(b) Subtract out the constant component of $spd[n]$, and then generate a MATLAB plot of the amplitude spectrum of $spd[n]$.

(c) From your result in Part (b), verify that the frequency of the dominant cyclic component has the expected value.

(d) Determine a smoothed version of $spd[n]$, using only the constant component of $spd[n]$ and the dominant cyclic component of $spd[n]$. Generate a MATLAB plot of the smoothed version of $spd[n]$.

(e) Generate a MATLAB plot consisting of both $spd[n]$ and the smoothed version of $spd[n]$. How well does the smoothed version fit the data? Discuss your conclusions regarding the fit.

4.21. Using the method given in Section 4.5, determine a smoothed version of the closing price $c[n]$ of QQQQ for the 60-business-day period April 27, 2004, up to July 22, 2004.